method for elliptic equations

Here we consider the details of the numerical method for the case of the conformal factor ψ only. The other elliptic equations are handled in a similar approach. The material below is a bit more specific for ψ than that I suggested you look at on my web page Thursday night. The form of the equation that is finite differenced for ψ has somewhat simpler coefficients due to the multiplication of the whole equation by a common factor, as mentioned below. Also, we actually solve for X and require that it vanish at infinity. The notes which follow were written by my collaborator, David Garfinkle. I have merely added a few coments pertinent to my making the stuff available to you for the purpose of deciding whether PETSc is the right set of tools...

I hope these notes help put some needed flesh on the bones of the previous notes.

The initial value equation for the conformal factor has the form

$$\frac{\partial^2 \psi}{\partial r^2} \; + \; \frac{1}{r} \, \frac{\partial \psi}{\partial r} \; + \; \frac{\partial^2 \psi}{\partial z^2} = F \psi$$

for some "source" F. We change variables to $(\tilde{r}, \tilde{z})$ defined by $r = \tan \tilde{r} \quad z = \tan \tilde{z}$. We also introduce $X = \psi - 1$ so that the new variable X vanishes at infinity. The equation then becomes

$$\frac{\cos^3 \tilde{r}}{\sin \tilde{r}} \, \partial_{\tilde{r}} (\sin \tilde{r} \, \cos \tilde{r} \, \partial_{\tilde{r}} X) \; + \; \cos^2 \tilde{z} \, \partial_{\tilde{z}} \left(\cos^2 \tilde{z} \, \partial_{\tilde{z}} X \right) = F(1 + X)$$

One can write this as

$$\partial_{\tilde{r}} (B \partial_{\tilde{r}} X) \; + \; \partial_{\tilde{z}} (C \partial_{\tilde{z}} X) \; - \; EX = E$$

where

$$B = \frac{\sin \tilde{r} \cos \tilde{r}}{\cos^2 \tilde{z}}$$

$$C = \frac{\cos^2 \tilde{z} \, \sin \tilde{r}}{\cos^3 \tilde{r}}$$

$$E = \frac{F \, \sin \tilde{r}}{\cos^3 \tilde{r} \cos^2 \tilde{z}}$$

We now consider how to finite difference the elliptic equation. Introduce for any continuous function $H(\tilde{r}, \tilde{z})$ the discrete quantities $H^i_j = H([i - 1.5]dr, [j - 1.5]dz)$. Note that $i = 1$ is a "ghost zone" not in the physical space and with its value to be set by boundary conditions, not field equations. In our case, since X is even in both $\tilde{r}$ and $\tilde{z}$, the appropriate boundary condition is $X^1_j = X^2_j$ and $X^i_1 = X^i_2$. The outer boundary also uses ghost zones in the following sense: $i = nr$ corresponds to $\tilde{r} = (\pi/2) + (dr/2)$. Since X vanishes at spatial infinity the corresponding boundary conditions are $X^{nr}_j = -X^{nr-1}_j$ and $X^i_{nz} = -X^i_{nz-1}$.

Now consider the elliptic equation itself. We discretize the first term as follows: (omitting the subscript j which is the same for all terms)

$$\partial_{\tilde{r}}(B\partial_{\tilde{r}}X) \rightarrow \frac{1}{dr}\ \left[B^{i+1/2}(\partial_{\tilde{r}}X)^{i+1/2}\ -\ B^{i-1/2}(\partial_{\tilde{r}}X)^{i-1/2}\right]$$

$$= \frac{1}{dr^2}\ \left[\frac{1}{2}\,(B^i + B^{i+1})\,(X^{i+1} - X^i)\ -\ \frac{1}{2}\,(B^i + B^{i-1})\,(X^i - X^{i-1})\right]$$

The same method is used for the other term. This allows us to put the elliptic equation in the form $AX = b$ where

$$(AX)^i_j = \frac{1}{2}\left[(B^i_j + B^{i+1}_j)(X^i_j - X^{i+1}_j)\ +\ (B^i_j + B^{i-1}_j)(X^i_j - X^{i-1}_j)\right]$$

$$+\ \frac{1}{2}\,(dr/dz)^2\ \left[(C^i_j + C^i_{j+1})(X^i_j - X^i_{j+1})\ +\ (C^i_j + C^i_{j+1})(X^i_j - X^i_{j+1})\right]\ +\ (dr)^2\,E^i_j X^i_j$$

and where

$$b^i_j = -\,(dr)^2\,E^i_j$$

In the definition of this operator, one considers X^i_j and $(AX)^i_j$ to be defined only on the physical zones. Then in order to compute AX one defines X on the ghost zones as well using boundary conditions and then uses this extra information to compute AX on the physical zones. For any two "vectors" X and V define the inner product

$$\langle V, X\rangle = \sum_{i=2}^{nr-1}\ \sum_{j=2}^{nz-1}\ V^i_j\ X^i_j$$

Then with this definition of inner product one can show that A is self adjoint in the sense that

$$\langle V, AX\rangle = \langle AV, X\rangle$$

Therefore the conjugate gradient method can be applied to this system.