The Elliptic Partial Differential Equations in code coordinates

We now introduce new coordinates $(\tilde{r},\tilde{z})$ given by $r = \tan\tilde{r}$ and $z = \tan\tilde{z}$. We keep most of the variables given before but introduce two new ones: $\tilde{S} \equiv S/\cos\tilde{r}$ and $\tilde{W} \equiv W/\cos\tilde{r}$. Note that this implies $rS = \sin\tilde{r}\tilde{S}$ and $rW = \sin\tilde{r}\tilde{W}$

What follows are the explicit forms of the three elliptic pde's we solve each time step using Neumann preconditioned Conjugate Gradient. The variables of interest as far as the elliptics are concerned are ψ,α,β^r, and β^z. The elliptic equations given below are solved in the order given.

The initial value equation for ψ becomes

$$\frac{\cos^3\tilde{r}}{\sin\tilde{r}}\,\partial_{\tilde{r}}(\sin\tilde{r}\,\cos\tilde{r}\,\partial_{\tilde{r}}\psi) \;+\; \cos^2\tilde{z}\;\partial_{\tilde{z}}\left(\cos^2\tilde{z}\;\partial_{\tilde{z}}\psi\right)$$

$$= -\frac{\psi}{4}\;\left[\cos^2\tilde{r}\;\partial_{\tilde{r}}\left(\cos^2\tilde{r}\partial_{\tilde{r}}(\sin\tilde{r}\tilde{S})\right) \;+\; \sin\tilde{r}\cos^2\tilde{z}\;\partial_{\tilde{z}}\;\left(\cos^2\tilde{z}\;\partial_{\tilde{z}}\tilde{S}\right)\right] \quad .$$

The equation for the lapse becomes

$$\frac{\cos^3\tilde{r}}{\sin\tilde{r}}\,\partial_{\tilde{r}}(\psi^2\sin\tilde{r}\cos\tilde{r}\partial_{\tilde{r}}\alpha) \;+\; \cos^2\tilde{z}\partial_{\tilde{z}}\left(\psi^2\cos^2\tilde{z}\;\partial_{\tilde{z}}\alpha\right)$$

$$= \alpha\psi^6 e^{2\sin\tilde{r}\tilde{S}}\;\left[\frac{2}{3}\left(U^2 \;+\; \sin^2\tilde{r}\tilde{W}^2 \;+\; U\sin\tilde{r}\tilde{W}\right) \;+\; 2({K^z}_r)^2 \;+\; \frac{1}{4}\;H\right] \quad .$$

The equations for the shift become

$$\cos^2\tilde{r}\;\partial_{\tilde{r}}\left(\cos^2\tilde{r}\partial_{\tilde{r}}\beta^r\right) \;+\; \cos^2\tilde{z}\;\partial_{\tilde{z}}\left(\cos^2\tilde{z}\partial_{\tilde{z}}\beta^r\right)$$

$$= 2\cos^2\tilde{z}\;\partial_{\tilde{z}}(\alpha{K^z}_r) \;-\; \cos^2\tilde{r}\;\partial_{\tilde{r}}(\alpha U) \quad ,$$

$$\cos^2\tilde{r}\;\partial_{\tilde{r}}\left(\cos^2\tilde{r}\partial_{\tilde{r}}\beta^z\right) \;+\; \cos^2\tilde{z}\;\partial_{\tilde{z}}\left(\cos^2\tilde{z}\partial_{\tilde{z}}\beta^z\right)$$

$$= 2\cos^2\tilde{r}\;\partial_{\tilde{r}}(\alpha{K^z}_r) \;+\; \cos^2\tilde{z}\;\partial_{\tilde{z}}(\alpha U) \quad .$$