Chapter 12

Applications of the Special Theory

I. Introduction

We take the postulates discussed in Chapter 11 to be confirmed by the experimental tests we outlined there, not to mention the many indirect tests of the postulates in concert with other assumptions which have been made literally billions of times over the last 75 years. It is our task in this chapter to explore in some detail some of the most important ramifications of the postulates. This exploration will take the form of a series of analyses of various issues exposed by a straight forward reading of the physical content of the postulates.

Our treatment will introduce an approach which we believe is useful, for the description of all of the applications of the Special Theory. We assume each inertial observer has a clock and the ability to send and receive light signals. With these two properties each observer can, by making maximal use of just these experiences on his world line, discover the motion dependence of the time and spatial interval between events off his world line. The use of a small number of constructs satisfies another need as well. This technique allows us to develop the applications of the postulates in as simple a manner as possible, without the use of sophisticated mathematics.

First we will discuss the concept of simultaneity. This is not a new issue for us. In Chapter 9 we found that for all spacetime models from Aristotelian through Newtonian a concept of “simultaneity slice” forms an intrinsic part of the models. It is with Einstein’s Special Theory that this concept will be necessarily changed. We will explain in the present chapter how a new concept of simultaneity will naturally rise from the ashes of the Galilean spacetime simultaneity slice, given the validity of Postulate II.

Second we will explain how the new concept of simultaneity is sensitive to the relative motion of pairs of inertial observers, in contrast to the observer independent of the simultaneity of events in the Galilean spacetime.

After motivating the concept of relativistic simultaneity we will analyze the major effects of the Special Theory: time dilation and length contraction.

II. Galilean Simultaneity

A fundamental issue which was faced by Einstein in his 1905 paper on the Special Theory of Relativity was the concept of simultaneity of events. As a means of orienting the reader, let us recall the basic ingredients of the concept of simultaneity in the Galilean Spacetime model. This model has a universal time function which serves to mark the time component of an event's description in a way that is the same for all observers. The t = constant slices denote the set of events simultaneous with a given event. Let us ask how a neo-Galilean observer G might go about constructing such a subset of events. One way in which this might be done is to have G send out some sort of probe signals from his worldline to various events of interest. The signals would then be reflected back to G, where their arrival times would be compared with the times they were sent.

This procedure requires that some choice be made for the "probing stuff". The observer G could advantageously make use of the fact that there is no upper limit to the velocity with which objects can move. This implies that <u>instantaneous communication</u> between a given event and other, arbitrarily distant events would be possible in principle. Consequently, the procedure would go as follows: send out an infinitely fast signal and receive its reflection instantaneously. Any number of events can be collected together by this procedure, all simultaneous with a given event.

What are the properties of the class of events thus constructed? First, all events in the class have the <u>same</u> reading on the universal clock, which ticks away absolute time. Second, the <u>distance</u> between the given, reference event and any other event in the class is arbitrary. This is because of the infinite velocity available for the sent and the reflected signals. An event spatially nearby would be reached by the outgoing probe signal at the same instant as would an arbitrarily distant event. Third, the state of motion of the given observer has no effect on his assessment of simultaneity, as it is a corner stone of the Galilean spacetime model that the uniform relative motions of different observers has no observable consequences. Consequently, suppose a G_1 observer finds that some event e is simultaneous with an event g_1 on his worldline. Another observer O_2, uniformly moving with respect to G_2, will find the event e to be simultaneous with an event g_2 on his worldline. The motion independence of any objective assessments that these two observers make implies that G_1 and G_2 will both find the events e, g_1 and g_2 to be simultaneous. We illustrate this state of affairs in Fig. 12.1.

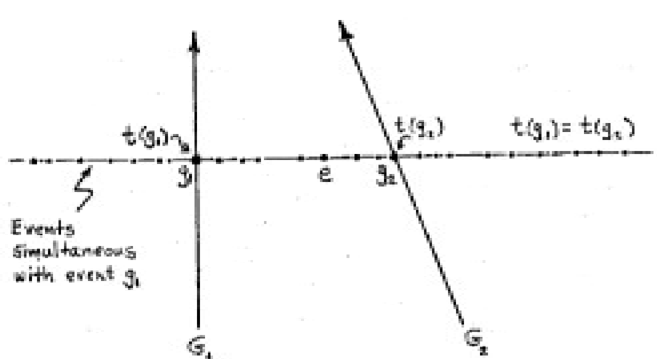


Fig. 12.1

The same property holds for a given observer's assessment of the simultaneity of <u>two</u> events e_1 and e_2 off his worldline. For, as shown in Fig. 12.2, if e_1 and e_2 are simultaneous with g_1, the same relation of simultaneity will be found by G_2.

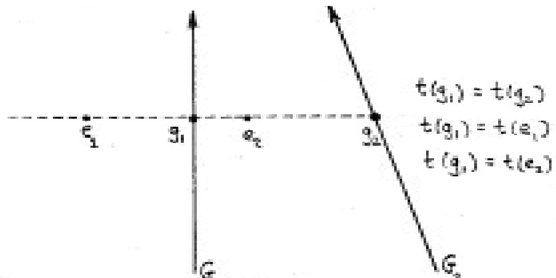


Fig. 12.2

From Einstein's point of view in the early l900's, one thing stands out as a problem, namely, the role of the probe signal in our characterization of simultaneity. One must consider the plausibility of the physical possibility of sending information at infinite velocity. There is no known material which can or does achieve infinite velocity. Thus, Einstein's problem was to identify an appropriate signal material which could be used to determine a physically realizable definition of simultaneity. The previous considerations suggest that one find a material possessing the fastest possible signal velocity. By the early l900's it was well known that the velocity of light was very large compared to the velocity of any other object. Light thus seems to be the prime candidate for the signalling substance. From this point on in the development of spacetime models, light assumes a pre-eminent position, culminating in the unification of electromagnetic and mechanical theories.

<u>III. Time and Distance Measurements Using Light Clocks and Signals</u>

In the sequel we will discuss time measurements which will be made by a single observer. Thus it is well to have an economical characterization of the concept of clock. We give a characterization which makes use of light and a minimum of additional constructs, such as mirrors.

In the following paragraphs we discuss an example of a clock that is constructible in principle and turns out to be useful for several discussions in relativity theory thought experiments. This is the so-called "light clock". It is constructed by placing two mirrors some distance apart. Attached to one of the mirrors is a source of light pulses. Think of a laser pulsing away, emitting bunches of photons. These pulses travel from one mirror to the other and then bounce back to the first mirror. We define "one tick" of such a clock to be that time required for a single round trip. In Fig. 12.3 we show a very schematic diagram of such a clock (12.3a) and the spacetime diagram (12.3b) of several ticks.

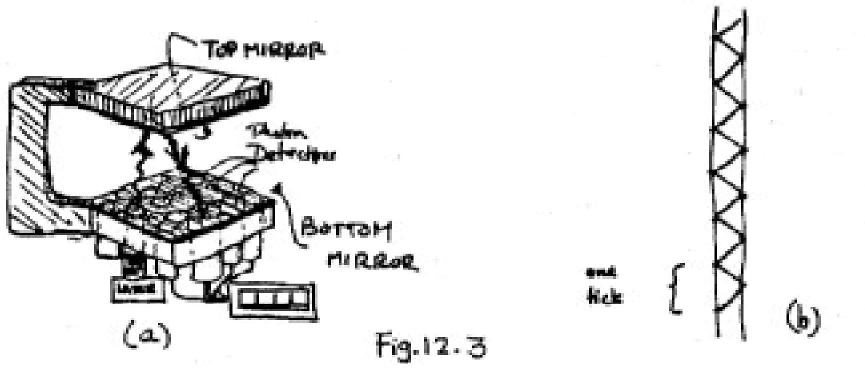


Fig.12.3

This example provides us with a kind of clock that is very simple in principle and uses light in an essential way. Since the speed of light is unaffected by the motion of the source it follows that any such "light clock" will be as immune as possible from effects that depend upon motion

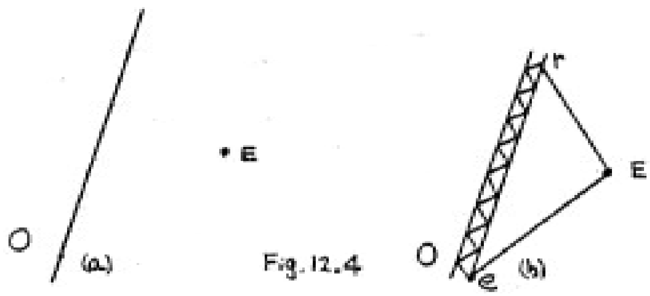
Another kind of clock is provided by the oscillations of electro-magnetic waves emitted when an atom emits or absorbs a quantum of radiation. The structure of the atom is such that it can only exist in certain allowed energy states. When an atom in an energy state of energy $E_2 > E_1$ makes a transition form E_2 to E_1 it emits electromagnetic radiation whose frequency of oscillation is proportional to the difference between the two energies. The frequency of this wave is very precise. Thus if one measures the frequency f one in essence has a very accurate clock since the period of the wave is $T = 1/f$.

The two examples just discussed do not exhaust the possibilities. The point is that any observer can carry with him a device which possesses the required repetitive property we usually connote by "clock". Throughout our subsequent discussions in this chapter let us assume that each observer has his proper time measuring clock, the structure of which is also assumed to be sufficiently ideal so as to obviate any further discussion of any "mechanical" faults.

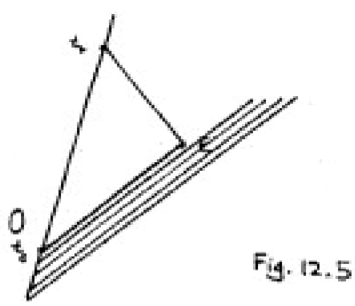
There is a second assumption that we find very convenient to make. This is that each observer has the ability to send and receive light signals.

We will assume, therefore, that every observer (a) has a light clock that measures proper time, and (b) has the ability to send and receive light signals. We turn to a discussion of further effects that are consequences of our acceptance of the postulates.

Suppose an observer, armed with his clock and light signals wishes to record the time of occurrence of an event not on his worldline. How could he proceed to record such an event given that he can not "walk" over and look at the event and his clock at the same instant? He must stay at some distinct space-time point and make the determination. In Fig. 12.4(a) we show the event of interest and the world line of the observer on a spacetime diagram.



Our observer must now decide how to define the time of occurrence of E. One procedure would be to have O emit a light signal at some instant of time, let it travel out in all directions and subsequently arrive at E. Of course, for the observer to eventually know that his light probe successfully “found” E, a light signal would have to be reflected back to O’s worldline. Hence the complete sequence would be as shown in Fig. 12.4(b). There we show the observer’s light clock, which starts ticking at event e, corresponding to the emission of the light signal and stops ticking when event r is reached as O records the received signal from E. The total local proper time on O’s clock is the number of ticks on his light clock multiplied by the time required for a complete single tick. Incidentally if the fact that e just happens to have been situated so that it just “hits” E seems a bit rigged we simply remark that the observer could be continuously probing his environment with light signals. The event E will then be intersected by some signal and then echoed back. This is shown pictorially in Fig. 12.5.



We are ready to specify in detail how an observer O determines the time and the spatial distance of events that are not on his worldline. The idea is really very simple to describe. O emits a light signal at some time as measured on his local clock. The signal travels to the event E, is immediately reflected back to O’s worldline and is detected by O at some subsequent time. Let t_e be the time that the light signal is sent and t_r be the time that the return signal is received. The spacetime diagram of this seqeunce of events is shown in Fig. 12.6.

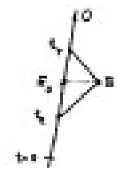


Fig. 12.6

We define the <u>time of occurrence</u> of the event E to be the time at which the first light signal was sent plus half of the total time required for the probe signal to go out to E and return. In symbols this reads:

$$t_E = t_e + 1/2\ (t_r - t_e)$$

or

$$t_E = 1/2\ (t_E + t_r) \qquad (12.1)$$

Next, we define the spatial distance from our position to E. We simply note that if the elapsed time for the probe signal's trip is $t_r - t_e$ then the <u>distance</u> to E is the speed of light times 1/2 this time or

$$d_E = c\ 1/2\ (t_r - t_e)$$

Recall, however, that we have adopted dimensions which make c = 1. Thus

$$d_E = 1/2\ (t_r - t_e) \qquad (12.2)$$

Using this technique any observer can spatially and temporally order all those events which he can reach (or which can reach him) by light signals.

<u>IV. Definition of Relativistic Simultaneity</u>

We can now develop a definition of relativistic simultaneity as follows. Let us identify an event E_o on the worldline of an observer.

To construct the set of events simultaneous with the event E_o the observer sends and subsequently receives a set of N light signals to various events off his worldline. Let us label the i th such pair of sent and received signals by giving the two times $t_s^{(i)}$, $t_r^{(i)}$. From our definition of the time of occurrence, there is a time <u>on</u> the worldline of the observer which coincides with the event intersected by the i th pair. This time is

$$t^{(i)} = \frac{1}{2}(t_s^{(i)} + t_r^{(i)})$$

It will not in general be the case that the time $t^{(i)}$ coincides with the time of the chosen event E_o on the observer's worldline. Thus, many of the probe signals will correspond to other events on the observer's worldline. So be it. We sift through the set of probe signals, calculate the time of occurrence for each and keep those which pass the test: $t^{(i)} = t_{E_p}$. The set of events constructed in this way defines what we shall call the relativistic simultaneity slice. This procedure may be shown pictorially as follows.

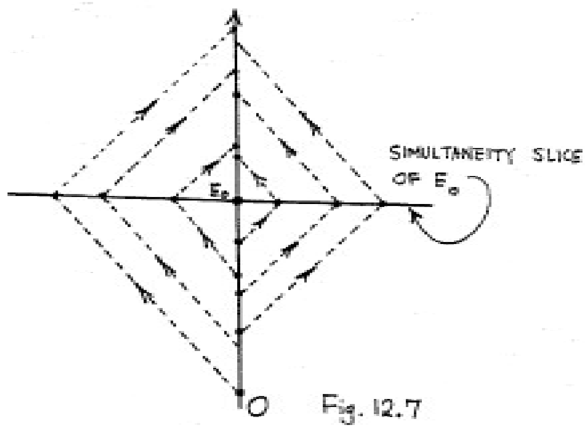


Fig. 12.7

Note that the finiteness of the velocity of light forces us to send probe signals out in advance of the event E_o and to receive the reflected probe signals after the event E_o. Consider a probe signal which was sent out very much earlier than the event E_o. The observer will receive the return signal much later than the event E_o. Such a pair t_s, t_r will have the property that $t_{E_0} >> t_s$ *and* $t_r >> t_{E_0}$. Consequently $t_r >> t_s$ Thus $t_r - t_s$ is a large real number. Recall that the spatial distance to a given event off the observer's worldline is $d_E = 1/2\ (t_r - t_s)$. We conclude that the event intersected by this very early sent, 'very late received' pair of signals is necessarily at a large spatial distance from the observer.

V. The Relativity of Relativistic Simultaneity

Recall that the simultaneity of events in the Galilean spacetime was independent of the relative state of motion of two inertial observers, as illustrated in Fig. 12.1. The simultaneity of events for relativistic observers, however, is dependent on their relative state of motion. That is, given that one inertial observer finds a set of events off his worldline which are simultaneous (the simultaneity slice), another inertial observer who is moving relative to the first will necessarily find these same events not to be simultaneous.

In order to see that relativistic simultaneity is relative we construct an example. In the previous section we discussed the construction of the set of events in spacetime which would have been simultaneous with a given event on the worldline of an observer (see Fig. 12.7). Among those events let us single out two special ones E_1 and E_2 for further study. These two have additional properties that they are the same spatial distance from E_o and light signals from E_1 and E_2 arrive at 0's worldline simultaneously. We illustrate these in the following figure.

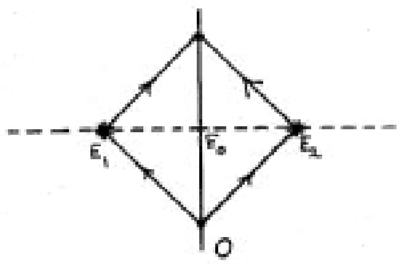


Fig. 12.8

Consider another inertial observer $\bar{0}$ who receives information about the three events E_o, E_1, and E_2. We arrange matters so that O crosses O's worldline just at the event E_o. The question is: will $\bar{0}$ experience E_1 and E_2 at the same time as measured by his clock that he carries with him? We claim that the answer is clearly NO! To see this we just have to construct the spacetime diagram of the situation described above and examine the times of reception to the light signals from E_1 and E_2. This is shown as follows.

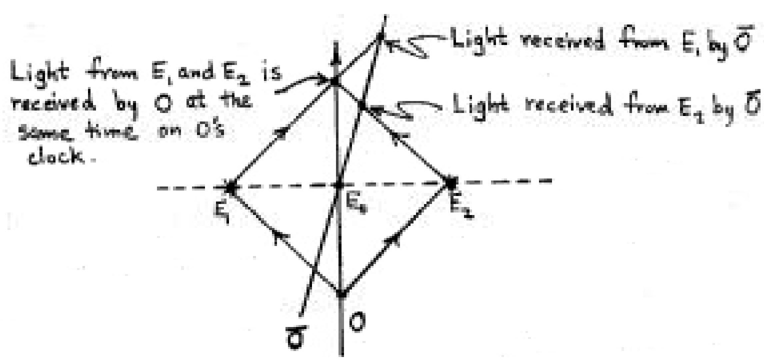


Fig. 12.9

The light from E_2 arrives at the worldline of $\bar{0}$ before the light from E_1. Consequently $\bar{0}$ would say that E_2 occurred before E_1, while 0 would say the two events were definitely simultaneous with E_o.

We conclude that, in general, two completely equivalent, but relatively moving inertial observers will not find the same collection of events to be simultaneous in the relativistic simultaneity sense. In the special relativity case this observer dependence of the "simultaneity slice" implies that the "slicing up" of spacetime by a given observer will not be universal. Different observers will not agree on the collection of events simultaneous with each observers local clock. This means that the universal time function of the Galilean spacetime is no longer applicable. In essence this implies the demise of absolute time.

VI. Time Dilation and Length Contraction

Our next task is to analyze some of the relationships between sequences of events off O's worldline. To make the treatment as simple as possible, we first consider the problem of what can be learned about sequences of events that are on the worldline of a particle that is moving with respect to O's worldline with some non-zero velocity. Let us arrange it so that the other moving particle's worldline crosses O's at some time on O's clock. Further let it be arranged so that the two local clocks, one O's and the other attached to the moving particle, to be labelled $\bar{0}$. Both O and $\bar{0}$'s clocks are reset to t = $\bar{t}$ = 0 at the point of crossing. Now at some instant after the crossing let O emit two light signals, one at time t_1 and the other at time t_2. Let the difference $t_2 - t_1 = T_o$. The question is: what time interval will be measured by the moving observer $\bar{0}$ attached to the particle? The situation is shown in Fig. 12.10.

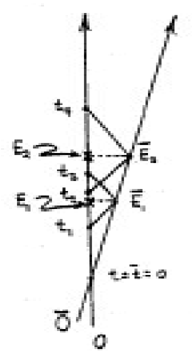


Fig. 12.10

The signal sent at t_1 will be received at some event $\overline{E_1}$, while the signal sent at t_2 will be received at some event $\overline{E_2}$, both events on the worldline of the moving observer $\bar{0}$. Let these two light signals be echoed as soon as sent so that they arrive at O's worldline at the times t_3 and t_4 respectively. Now from our definitions above, we can compute the time of occurrence of the events E_1 and E_2 which are simultaneous with $\overline{E_1}$ and $\overline{E_2}$. We have

$$t_{E_1} = \frac{1}{2}(t_1 + t_3)$$

$$t_{E_2} = \frac{1}{2}(t_2 + t_4)$$

Then the interval between E_2 and E_1 measured by O is given as follows:

$$t_{E_2} - t_{E_1} = \frac{1}{2}(t_2 + t_4) - \frac{1}{2}(t_1 + t_3) = \frac{1}{2}(t_2 - t_1) + \frac{1}{2}(t_4 - t_3)$$

The difference $t_4 - t_3$, measured by O's local clock, will be proportional to T because these clock readings are real numbers. Since the clock's readings are real numbers, there exists another real number λ such that $t_4 - t_3 = \lambda(t_2 - t_1)$. Thus, it follows that

$$t_{E_2} - t_{E_1} = \frac{1}{2} T_0 + \frac{1}{2} \lambda T_0 = (\frac{1+\lambda}{2}) T_0 \equiv \sigma T_0 \tag{12.3}$$

for some real number σ. That is, the interval between the events on O's worldline simultaneous with the events $\overline{E_1}$ and $\overline{E_2}$ is proportional to the interval between the two signals that were sent. This argument shows that there is a real number which captures something of the motion dependence of a moving observer's record of the relative motion. We will, in short order, determine how σ depends on the relative motion.

Let us pause to mention the basic idea behind this method. Suppose we have a source of waves and an observer who detects the waves emitted by the source. Consider two situations. First, let the source of waves and the observer be relatively at rest. The source emits waves with some characteristic wavelength. We indicate the situation in Fig. 12.11,

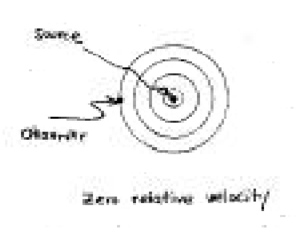


Fig.12.11

In this situation the observer measures the distance between successive crests of the waves and finds the wavelength (the distance between successive crests) to be identical to the wavelength produced by the source.

Contrast the above situation with that in which the source of waves moves with some velocity, say away from the stationary observer. What will the observer O now measure for the wavelength? To see the answer qualitatively consider the following diagram.

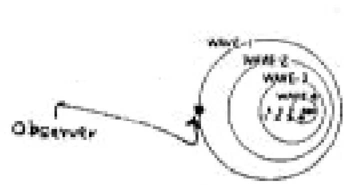


Fig. 12.12

The observer measures the distance between successive crests to be <u>larger</u> than he found when the source was stationary relative to the observer.

This is because the source is running away from 0. The source emits waves just as it did before relative motion was introduced. However, during the time a given crest travels out from the source and another crest is emitted, the source has translated a distance to the right. The second crest thus has to travel further than the first to get to O's position. Hence O observes a longer wavelength. Now there is a reciprocal relation between wavelength and the frequency of the source: frequency = (speed of propagation of wave)/(wave-length). Thus a longer wavelength implies a <u>lower</u> frequency. Finally, T, the <u>period</u> of the wave, is defined to be T = 1/frequency. Consequently the period is longer for a lower (i.e. smaller) frequency. This result is usually referred to as the Doppler effect. If we turn the situation around and have the source move <u>toward</u> the observer, the observed wavelength will be shorter due to the "crowding" of successive crests in the direction of motion. Hence in this case the frequency will be higher and the period shorter. These intuitive, "common sense" pictures are in essence incorporated in the Einstein definition of simultaneity and in the determination of the time of occurrence of events which are not on O's worldline.

Given the result of the above discussion, suppose we turn the situation around and let the moving observer emit two signals separated by a time interval T on his clock. Then if there is to be complete equivalence between the two observers we must require that the interval of the received signals be the same number σ times the sent interval. If this is not demanded then it would be possible to distinguish between the two observers. Postulate I forbids this. In Fig. 12.13 we show that two applications of the rule

$$T_{received} = \sigma T_{sent} \quad give \quad T = \sigma^2 T_0 .$$

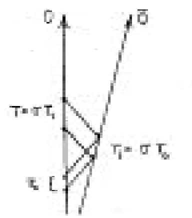


Fig.12.13

Given our discussion of the Doppler effect above we expect that there is some relative velocity dependence in σ. Our next task is to find out just how this dependence is manifested. To see this we consider a two-observer system like the above. This time we will work toward expressing the quantity σ in terms of the velocity of $\bar{0}$ with respect to O. Thus suppose O sends a light signal from 0 to $\bar{0}$ which arrives at event B_2. Let B_1 be the event which is the intersection of O's worldline with $\bar{0}$'s worldline. This sequence of events is shown in the following Fig. 12.14.

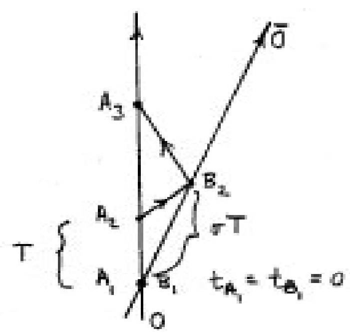


Fig. 12.14

Using the relation $T_{received} = \sigma T_{sent}$ we derive below a needed relation between $t_{A_1},\ t_{A_2},\ and\ \ t_{A_3}$.

$$t_{B_2} - t_{B_1} = \sigma(t_{A_2} - t_{A_1})$$

$$t_{A_3} - t_{A_1} = \sigma(t_{B_2} - t_{B_1}) = \sigma^2(t_{A_2} - t_{A_1})$$

(12.4)

The definition of the velocity of $\bar{0}$ is $V_{\bar{0}}$ = d/t as measured by 0. Using Eq. 12.1 and Eq. 12.2 and inspecting Fig. 12.14 we have for $V_{\bar{0}}$:

$$v_{\bar{0}} = \frac{\frac{1}{2}(t_{A_3} - t_{A_2})}{t_{A_2} - t_{A_1} + \frac{1}{2}(t_{A_3} - t_{A_2})} \tag{12.5}$$

.

Consider the quantity t_{A_3} - t_{A_3} that occurs in the above equation.

$$t_{A_3} - t_{A_2} = (t_{A_3} - t_{A_1}) + (t_{A_1} - t_{A_2}) = \sigma^2 (t_{A_2} - t_{A_1}) - (t_{A_2} - t_{A_1})$$

$$t_{A_3} - t_{A_2} = (\sigma^2 - 1)(t_{A_2} - t_{A_1}) \tag{12.6}$$

Using Eq. 12.6 we have

$$t_{A_2} - t_{A_1} + \frac{1}{2}(t_{A_3} - t_{A_2}) = \frac{(t_{A_2} - t_{A_1})}{2}\left[\sigma^2 + 1\right] \tag{12.7}$$

Now we are ready to simplify the denominator of Eq. 12.5. Substitute Eq. 12.6 into Eq. 12.5, with the result:

$$V = \frac{\frac{1}{2}(\sigma^2 - 1)\ (t_{A_2} - t_{A_1})}{\frac{(t_{A_2} - t_{A_1})}{2}\ (\sigma^2 + 1)}$$

$$V_{\bar{0}} = \frac{\sigma^2 - 1}{\sigma^2 + 1} \tag{12.8}$$

This expresses $V_{\bar{0}}$ in terms of σ. It is a two line calculation to invert to express in terms of V, with the result

$$\sigma = \sqrt{\frac{1 + V}{1 - V}} \tag{12.9}$$

Now V is restricted to be less than 1, corresponding to the vacuum speed of light. σ is 1 when V=0 and monotonically increases as V approaches 1.

Thus we have established the manner in σ which depends on the velocity of $\bar{0}$ with respect to 0. In addition we can see how our analysis is similar to a “radar ranging” method for measuring the velocity: (1) bounce two “radar” signals off the moving object, (2) measure the ratio of received to emitted time intervals between the two signals. This gives σ via the relation $T_{\text{received}} = \sigma T_{\text{sent}}$. (3) The velocity is then calculated via $v = (\sigma^2 - 1)/(\sigma^2 + 1)$.

Time Dilation

Consider two inertial observers A and B who have a non-zero relative velocity V. We arrange it so that A and B have identical clocks. The question we wish to pose and answer is: if A’s clock has an elapsed time T for one tick of his light clock, what will be the time for a tick of observer B’s light clock as measured by A? In order to answer the above question we set up the experiment illustrated in Fig. 12.13.

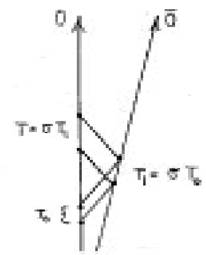


Fig. 12.13

The two tilted but parallel lines represent the world lines of the two mirrors of B’s light clock. One mirror coincides with B’s worldline, while the other mirror is at rest relative to B. The events B_1 and B_2 correspond to the completion of one tick of B’s clock. In order for A to know about the tick, he must emit the two probe signals at t_1 and t_3. These are received back respectively at t_2 and t_4. According to A, B_1 and B_2 occur at times t_{B_1} and t_{B_2} on A’s clock where:

$$t_{B_1} = \frac{1}{2}(t_1 + t_2)$$

$$t_{B_2} = \frac{1}{2}(t_3 + t_4)$$

The time interval between B_1 and B_2 as measured by A is given by

$$T_{1\,2\,A} = t_{B_2} - t_{B_1} = \frac{1}{2}[(t_3 + t_4) - (t_1 + t_2)] \qquad (12.10)$$

Let us refer to clock readings on B’s worldline by τ_1 *and* τ_2. Observer B sends a light signal to the mirror M_2 at some time τ_1 and receives it back y at time τ_2. As before we arrange it so that observers A and B start their clocks when their origins cross. Thus between crossing and the sending of the first signal by A a time t_1 has elapsed. It follows from our analysis of the relation between intervals sent and received that

$$\tau_1 = \sigma t_1 \qquad (12.11)$$

Consequently $\sigma(\sigma t_1)$ is the interval received by A at t_2. A similar remark holds for t_4. Thus we can state the two relations

$$t_2 = \sigma\tau_1 = \sigma^2 t_1 \tag{12.12}$$

$$t_4 = \sigma\tau_2 = \sigma^2 t_3 \tag{12.13}$$

On B's worldline let us call the elapsed time between B_1 and B_2 $T_{12/B}$. In terms of τ_1 and τ_2, and using the relations in Eq. 12.12, Eq. 12.13 we have for $T_{12/B}$:

$$T_{12/B} = \tau_2 - \tau_1 = \frac{1}{\sigma} t_4 - \sigma t_1 \tag{12.14}$$

Our analysis of the tick of B's clock as experienced by A has thus led to the following set of four equations in the observable quantities.

$$T_{12/A} = \frac{1}{2}\left[(t_3 + t_4) - (t_1 + t_2)\right] \tag{12.15}$$

$$T_{12/B} = \frac{1}{\sigma} t_4 - \sigma t_1 \tag{12.16}$$

$$t_2 = \sigma^2 t_1 \tag{12.17}$$

$$t_4 = \sigma^2 t_3 \quad or \quad t_3 = \frac{1}{\sigma^2} t_1 \tag{12.18}$$

We want to solve for $T_{12/A}$ in terms of the velocity of B with respect to A and the interval $T_{12/B}$ for the tick on B's clock. Substitute Eq. 12.17 and Eq. 12.18 into Eq. 12.15:

$$T_{12/A} = \frac{1}{2}\left[\left(\frac{1}{\sigma^2} t_4 + t_4\right) - (t_1 + \sigma^2 t_1) \right]$$

$$= \frac{1}{2}\left[t_4\left(\frac{1+\sigma^2}{\sigma^2}\right) - t_1(1+\sigma^2) \right]$$

$$= \frac{1}{2}(1+\sigma^2)\ \left[\frac{t_4}{\sigma^2} - t_1\right]$$

Next solve Eq. 12.16 for t_4:

$$t_4 = \sigma T_{12/B} + \sigma^2 t_1 \tag{12.19}$$

Finally the expression for $T_{12/A}$ becomes

$$T_{12/A} = \frac{1}{2}(1+\sigma^2)\ \frac{\left[\sigma T_{12/B} + \sigma^2 t_1\right]}{\sigma^2} - t_1$$

$$= \frac{1}{2}(1+\sigma^2)\frac{T_{12/B}}{\sigma} \tag{12.20}$$

From the Eq. 12.9 we can show that $(1+\sigma^2)/2\sigma = (1-v^2)^{-\frac{1}{2}}$. Hence we find that

$$T_{12/A} = \frac{T_{12/B}}{\sqrt{1-v^2}} \tag{l2.21}$$

Now $(1 - v^2)$ is always smaller than 1 for $v<1$. Thus $(1-v^2)^{-\frac{1}{2}}$ is always greater than 1. Consequently $T_{12/A}$ is always greater than $T_{12/B}$. Therefore the Special Theory of Relativity predicts that <u>moving clocks run slow</u>. This is the Time dilation effect and forms one of the most startling consequences of the postulates.

This effect has been conclusively tested in several experiments. The most recent was an experiment in which sub-atomic particles, mu-mesons, were produced in a high-energy accelerator and stored there in a “storage ring”. The particles had a velocity of 99.652% of the vacuum speed of light. This experiment was performed by measuring the interval of time the mu-mesons existed before they underwent their normal decay into other particles. The high-velocity mu-mesons were found to live longer than mu-mesons that decayed at rest relative to the apparatus. The result of these studies was that Einstein's time dilation formula was verified.

Length Contraction

Let us consider the two observers A and B again. Let A and B possess two identical light clocks. Thus the distance between the two mirrors is measured to be the same when A and B are relatively at rest. Now, however, our question has changed to: what is the length experienced by A of the distance between the two mirrors that make up B's light clock? We will use the same set-up as before. For the calculation of the length contraction effect we focus on a different group of events. In the time dilation case we wanted to compare observer A's measurement of the ticking of B's clock as compared to A's clock. Here we want to compare the distance determinations made by A and B of the distance between two points at rest with respect to B. This we do, remembering to consistently use the Einstein definition of simultaneity. Now in order to make a length determination the observer must make the measurement utilizing events that occur at the same time in the observer's frame of reference. We show the events of interest in the following figure.

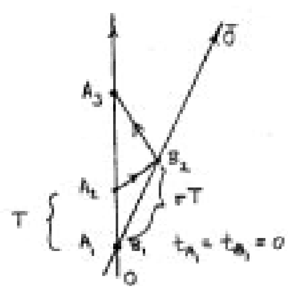


Fig. 12.14

At time t_1 A sends a pulse to intersect the far mirror of B's light clock at B_1 and receives the return signal at t_4. Observer A sends another pulse at time t_2 to intersect the near mirror (which also coincides with observer B's origin) at B_2 and receives the return signal at t_3. A crucial question that observer A must answer is: when should A send the two signals so that A sees B_1 and B_2 as simultaneous. Well, from our previous experience we can immediately state the answer. A will see B_1 and B_2 as simultaneous just in the case

$$\frac{1}{2}(t_1 + t_4) = \frac{1}{2}(t_2 + t_3) \tag{12.22}$$

Next we state the relations between various sent and received time intervals derived from a straightforward application of our basic rule Eq. 12.3.

$$t_3 = \sigma(\sigma t_2) = \sigma^2 t_2 \tag{12.23}$$

Now observer B can measure the length between the two mirrors of his clock by emitting the light signal that arrives from A at τ_1 receiving it at some time, τ_2, later. We refer to the send and receive times as T_1 and T_2. Then, the time required for the signal to go just to the far mirror will be the time $(T_2 - T_1)/2$ multiplied by the speed of light, which is 1 in our system of units. This gives for B's measurement of the length

$$L_B = \frac{1}{2}(T_2 - T_1) \tag{12.24}$$

Using our rule Eq. 12.3 again this relation becomes

$$L_B = \frac{1}{2}\frac{(t_4 - \sigma^2 t_1)}{\sigma} \tag{12.25}$$

The distance ℓ between B's mirrors as measured by A will be the difference between A's measurement of the position of the mirrors. In terms of our basic definition of distance in Eq. 12.2 we have

$$\ell = \frac{1}{2}(t_4 - t_1) - \frac{1}{2}(t_3 - t_2) \tag{12.26}$$

What follows are the manipulations necessary to take these relations and solve for the length ℓ in terms of L and any other relative velocity dependence that may be present. First we collect together all four relations.

$$\frac{1}{2}(t_1 + t_4) = \frac{1}{2}(t_2 + t_3) \tag{12.27}$$

$$t_3 = \sigma^2 t_2 \tag{12.28}$$

$$L = \frac{1}{2}(t_4 - \sigma^2 t_1)/\sigma \tag{12 29}$$

$$\ell = \frac{1}{2}(t_4 - t_1) - \frac{1}{2}(t_3 - t_2) \tag{12.30}$$

Our next move is to substitute Eq.12.28 into Eq. 12.27 and simplify. The result is

$$\frac{1}{2}(t_1 + t_4) = \frac{1}{2}(\sigma^2 + 1)t_2 \tag{12.31}$$

Next substitute Eq. 12.28 into Eq. 12.30 to obtain

$$\ell = \frac{1}{2}\left[(t_4 - t_1) - (\sigma^2 - 1)t_2\right] \tag{12.32}$$

Since we shall need it presently, solve Eq. 12.31 for t_2:

$$t_2 = \frac{1}{\sigma^2 + 1}(t_1 + t_4) \tag{12.33}$$

The next few lines are the result of substitution of Eq. 12.33 into Eq. 12.32

$$\begin{aligned}
\ell &= \frac{1}{2}\left[(t_4 - t_1) - \frac{(\sigma^2 - 1)}{(\sigma^2 + 1)}(t_1 + t_4)\right] \\
&= \frac{1}{2}\frac{\left[\sigma^2 t_4 + t_4 - \sigma^2 t_1 - t_1 - \sigma^2 t_1 + t_1 - \sigma^2 t_4 + t_4\right]}{\sigma^2 + 1} \\
&= \frac{1}{2}\frac{\left[2t_4 - 2\sigma^2 t_1\right]}{\sigma^2 + 1} = \frac{t_4 - \sigma^2 t_1}{\sigma^2 + 1}
\end{aligned} \tag{12.34}$$

Finally, from Eq. 12.29 we have

$$\ell = \frac{2\sigma L}{\sigma^2 + 1} \tag{12.35}$$

But the quantity $2\sigma/(\sigma^2 + 1)$ is just $(1-v^2)^{1/2}$. Consequently the final result is

$$\ell = L\sqrt{1 - v^2} \tag{12.36}$$

This is the famous <u>length contraction</u> formula. It states that there is a legitimate disagreement between A and B about the length between B's mirrors. Since $(1 - v^2)^{1/2}$ is always less than 1, we see that the length of the moving "ruler" is shorter than when at rest with respect to the observer.

Let us supply a numerical example, just to get some feeling for the velocities needed to have the length shrink to say 99% and then to 90% of the rest length. If we want ℓ/L to be .99 our formula above gives, when solved for v a velocity of B with respect to A of 4.2318 x 10^7 m/s. For a 10% reduction in the length we find that the velocity of B with respect to A would have to be 1.30764 x 10^8 m/s. At ordinary terrestial speeds (which almost never exceed 2000 miles/hr., or around 895 m/s) this difference is absolutely negligible. Thus we again learn that our limited experience with the properties of the world at very high speeds has once again given us a mistaken impression about the manner in which the distance between points might depend on other, relative attributes.